Inequality with infinite sum generated by decreasing positive sequences.
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Let D_1 be set of strictly decreasing sequences of positive real numbers with first term equal to 1. For any $\mathbf{x}_{\mathbb{N}} := (x_1, x_2, \ldots, x_n, \ldots) \in D_1$ prove that

$$\sum_{n=1}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}} \geq \frac{4}{9}$$

and find the sequence for which equality occurs.

Solution.

Let $S(\mathbf{x}_{\mathbb{N}}) = \sum_{n=1}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}}$ if series converges and $S_f(x_{\mathbb{N}}) = \infty$ if it diverges.

Let $\widetilde{D}_1 = \left\{ x_{\mathbb{N}} \mid x_{\mathbb{N}} \in D_1 \text{ and } S(x_{\mathbb{N}}) \neq \infty \right\}$. Since $\widetilde{D}_1$ isn't empty (because for for instance if $x_n = q^{n-1}, n \in \mathbb{N}$, where $q \in (0,1)$, we have $\sum_{n=1}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}} =$

$\sum_{n=1}^{\infty} \frac{q^{3(n-1)}}{q^{n-1} + 4q^n} = \sum_{n=1}^{\infty} \frac{q^{2(n-1)}}{1 + 4q} = \frac{1}{(1+4q)(1-q^2)}$ then

$\inf\{S(\mathbf{x}_{\mathbb{N}}) \mid \mathbf{x}_{\mathbb{N}} \in D_1\} = \inf\left\{S(\mathbf{x}_{\mathbb{N}}) \mid \mathbf{x}_{\mathbb{N}} \in \widetilde{D}_1\right\}$.

Let $S := \inf\left\{S(\mathbf{x}_{\mathbb{N}}) \mid \mathbf{x}_{\mathbb{N}} \in \widetilde{D}_1\right\}$. For any $\mathbf{x}_{\mathbb{N}} \in \widetilde{D}_1$ we have $S(\mathbf{x}_{\mathbb{N}}) = \sum_{n=1}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}} =$

$\frac{1}{1+4x_2} + \sum_{n=2}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}} = \frac{1}{1+4x_2} + x_2^2 \sum_{n=1}^{\infty} \frac{y_n^3}{y_n + 4y_{n+1}} = \frac{1}{1+4x_2} + x_2^2 S(\mathbf{y}_{\mathbb{N}})$, where $y_n := \frac{x_{n+1}}{x_2}, n \in \mathbb{N}$.

Since $\mathbf{y}_{\mathbb{N}} \in \widetilde{D}_1$ ($1 = y_1 > y_2 > \ldots > y_n > \ldots$ and $S(\mathbf{y}_{\mathbb{N}}) = \frac{S(\mathbf{x}_{\mathbb{N}})}{x_2^2} - \frac{1}{1+4x_2}$) then

$S(\mathbf{y}_{\mathbb{N}}) \geq S$ and, therefore, $S(\mathbf{x}_{\mathbb{N}}) \geq \frac{1}{1+4x_2} + x_2^2 S \Rightarrow S \geq \frac{1}{1+4x_2} + x_2^2 S \Leftrightarrow$

$S \geq \frac{1}{(1+4x_2)(1-x_2^2)}$.

We will find $\mu := \max_{x \in (0,1)} h(x)$, where $h(x) := (1+4x)(1-x^2) = -4x^3 - x^2 + 4x + 1$.

Since $h'(x) = -12x^2 - 2x + 4 = -2(3x+2)(2x-1)$ then

$\mu = \max_{x \in (0,1)} h(x) = h\left(\frac{1}{2}\right) = \frac{9}{4}$ and, therefore, $S(\mathbf{x}_{\mathbb{N}}) \geq \frac{1}{\mu} = \frac{4}{9}$.

Since $S(\mathbf{x}_{\mathbb{N}}) = \frac{1}{(1+4q)(1-q^2)}$ for $x_n = q^{n-1}, n \in \mathbb{N}$, $q \in (0,1)$, then for $q = \frac{1}{2}$

we obtain $\sum_{n=1}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}} = \frac{1}{\left(1 + 4 \cdot \frac{1}{2}\right)\left(1 - \left(\frac{1}{2}\right)^2\right)} = \frac{4}{9}$.